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ORIGINAL PAPER

Probabilistic evolution approach for the solution of explicit autonomous ordinary differential equations. Part 2: Kernel separability, space extension, and, series solution via telescopic matrices

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Abstract This work focuses on the Kronecker power series solution of the explicit conical ODEs. This means that the Kronecker power series of the descriptive function vector of the ODEs has only zeroth, first and second Kronecker powers of the unknowns hence the only nonvanishing matrix coefficients are $\mathbf{F}_0$, $\mathbf{F}_1$ and $\mathbf{F}_2$. We focus on the cases where $\mathbf{F}_0$ also vanishes. These enable us to get and solve a two block term recursive ODE and the accompanying initial conditions. The resulting Kronecker power series' kernel can be expressed as a binary product whose first factor which in square matrix type and a second factor which is in purely rectangular matrix algebraic structure. The constancy adding space extension separates the temporal behavior of the kernel in a scalar first factor while the second factor is again in rectangular matrix structure. We also show that the definition and use of rectangular eigenvalue problem takes us to constant solution of the original ODEs.

Keywords Dynamical systems · Ordinary differential equations · Kronecker or direct products · Kronecker or direct power series · Spectral decompositions

1 Introduction

Probabilistic evolution approach (PEA) is a novel theory which may be applied to the solutions of the initial value problems of ordinary differential equations [1–33]. It relies on series expansions similar to Taylor series. This is the companion to the paper entitled

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"Probabilistic evolution approach for the solution of explicit autonomous ordinary differential equations, Part 1: Arbitrariness and equipartition theorem in Kronecker power series" appearing in the same issue.

Probabilistic evolution approach is not based on discretization. Discretization methods are powerful tools and are quite appropriate for programming in computers [34–46]. The method proposed here is based on Taylor series, or more generally Kronecker power series. Probabilistic evolution equation obtained from the original set of equations is an infinite-dimensional linear equation the exact solution of which may be given in a formal way. In order to obtain numerical results, series truncations should be performed which in turn truncate the vectors and matrices of solution of probabilistic evolution equation. As a final statement of this paragraph, probabilistic evolution approach is a global approximation method. For this reason it is not open to error accumulation coming from recursive nature of discretization techniques. However, this does not imply a strict convergence in the truncation approximants. Our recent efforts focus on getting sufficiently rapid convergence when it exists. We also study on how to converge or get analytic continuation within this context.

First order ODE sets with analytic descriptive functions (which do not contain derivatives of unknowns at the right hand side while the left hand side includes only one unknown's derivative) can be converted to an infinite linear set of first order autonomous and homogeneous ODEs with a constant infinite coefficient matrix. The accompanying initial conditions are also populated to an infinite set of initial conditions. All these can be accomplished by using a complete basis set of terms functionally depending on unknown functions such that a new ODE is constructed for each element of this set. The constant infinite coefficient matrix depends on only the functional structures of the descriptive functions. The infinite linear ODE set can be formally solved in an analytic form which expresses the solution as the image of infinite initial vector under an exponential matrix whose argument is the abovementioned infinite coefficient matrix multiplied by time. The exponential matrix describes the propagation of the system while its argument is related to the rate of the propagation, in other words, the evolution. We call the infinite coefficient matrix "Evolution Matrix".

If the starting point for the PEA equations is a single ODE then there is just a single unknown and the basis set is composed of natural number powers of the difference between the unknown function and the Taylor series expansion point on the real axis. In this case, the initial vector is composed of the natural number powers of the initial value of the unknown while the evolution matrix becomes having an upper Hessenberg form whose each diagonal is generated by just a single term which is in fact proportional to the Taylor series of the descriptive function. The term, generating the lower neighbor diagonal to the main diagonal, vanishes when the descriptive function has a zero at the expansion point. Then upper Hessenberg form turns out to be upper triangular form which facilitates the spectral analysis of the evolution matrix and generally a discrete spectrum can appear only. Otherwise a possibility for the existence of continuous spectrum may arise. We mostly avoid continuous spectrum which corresponds to the singularities. Hence all analyses for PEA until now have been intensified at the focus of triangular block cases.

If the target initial ODE set is composed of not just a single but more than one equations and accompanying initial conditions then the analysis conceptually remains the same but the formulae become more comprehensively complicated. This may necessitate the use of the many indices and multiple sums if the Taylor series expansion is used as the mathematical tool. This may be avoided, by introducing the Kronecker power series which enables us to use just a single index in the sums, with the aid of the vectors and matrices of ordinary linear algebra. Thus, the resulting structures in the ultimate infinite linear ODE set contain an evolution matrix which is again in an upper Hessenberg form but this time not in scalar elements, instead, in block elements. The initial vector of this case also takes a block form composed of the Kronecker power of the initial vector appearing in the original ODE set's accompanying initial impositions. These are the important differences immediately coming to mind in the case of more than one unknowns.

We can now recall the main conceptual lines of the probabilistic evolution approach (PEA) by focusing on the following set of equations

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{f}\left(\boldsymbol{\xi}(t)\right), \qquad \boldsymbol{\xi}(0) = \mathbf{a}_{in} \tag{1}$$

where all entities symbolized by boldface characters are assumed to be composed of n elements, all of which are temporally varying except the ones in $\mathbf{a}_{in}$. While $\boldsymbol{\xi}(t)$ stands for the unknown vector varying in time the vector valued function $\mathbf{f}$ is assumed to be explicitly known. On the other hand, $\mathbf{a}_{in}$ which specifies the initial value of the unknown vector is assumed to be given.

The direct (Kronecker) power expansion of the right hand side (descriptive function vector) can be explicitly written as follows

$$\mathbf{f}\left(\boldsymbol{\xi}\right) = \sum_{j=0}^{\infty} \mathbf{F}_j \mathbf{s}^{\otimes j} \tag{2}$$

where each term of the expansion has the product of a coefficient matrix and a direct power of the system vector. By definition, each element of the system vector denotes the difference between the corresponding independent variable and expansion point component.

$$\mathbf{s} \equiv \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \equiv \begin{bmatrix} \xi_1 - \xi_1^{(r)} \\ \vdots \\ \xi_n - \xi_n^{(r)} \end{bmatrix}. \tag{3}$$

The direct product of $\mathbf{s}$ with itself is $\left[s_1\mathbf{s}^T \ \ldots \ s_n\mathbf{s}^T \right]^T$ which can also be considered as the direct (or Kronecker) square of $\mathbf{s}$. Also, nth direct power of the system vector can be given through the recursive relation $\mathbf{s}^{\otimes n} = \left[s_1\mathbf{s}^{\otimes(n-1)^T} \ \ldots \ s_n\mathbf{s}^{\otimes(n-1)^T} \right]^T$. By convention, zeroth direct power of any vector is just 1 and the first direct power of a vector is the vector itself. Since (2) should have same type (that is, $n \times 1$) additive components at its both sides, the matrix coefficient of the jth direct power of $\mathbf{s}$ is of $n \times n^j$ type. So $\mathbf{F}_0$ is just an n element vector composed of the descriptive function values at the expansion point while the square matrix $\mathbf{F}_1$ is the Jacobian matrix of the descriptive functions with respect to the unknowns, evaluated at the expansion point. It is responsible for the stability issues of the system. $\mathbf{F}_2$ and all remaining higher power coefficients are horizontally rectangular matrices. $\mathbf{F}_j$'s type is $n \times n^j$.

Now, if we define

$$\mathbf{x}_j(t) \equiv \mathbf{s}^{\otimes j}, \qquad j = 0, 1, 2, \ldots \tag{4}$$

$$\mathbf{x}(t) \equiv \left[\mathbf{x}_0(t)^T \; \ldots \; \mathbf{x}_n(t)^T \; \ldots \; \right]^T, \tag{5}$$

$$\mathbf{E} \equiv \begin{bmatrix} \mathbf{E}_{0,0} & \cdots & \mathbf{E}_{0,j} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{E}_{j,0} & \cdots & \mathbf{E}_{j,j} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{6}$$

$$\mathbf{a} \equiv \begin{bmatrix} \left(\mathbf{a}_{in} - \boldsymbol{\xi}^{(r)}\right)^{\otimes 0} \\ \vdots \\ \left(\mathbf{a}_{in} - \boldsymbol{\xi}^{(r)}\right)^{\otimes j} \\ \vdots \end{bmatrix}^T \tag{7}$$

then we can write

$$\dot{\mathbf{x}}(t) = \mathbf{E}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{a}. \tag{8}$$

The formal solution of (8) can be expressed as follows

$$\mathbf{x}(t) = \mathrm{e}^{t\mathbf{E}}\mathbf{a} \tag{9}$$

where the exponential matrix stands for the system's propagator while $\mathbf{E}$, the system's evolution matrix, characterizes the temporal evolution rate of the system. The block elements of the evolution matrix can be given as follows

$$\mathbf{E}_{j,j+k-1} \equiv \sum_{\ell=1}^{j-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{F}_k \otimes \mathbf{I}_n^{\otimes j-1-\ell}, \qquad j, k = 0, 1, 2, \ldots \tag{10}$$

Kronecker power expansions containing only two nonzero terms of the infinite sum are very important because many explicit ODEs can be brought to this form by appropriate unknown redefinitions (even increasing their populations). Such structures facilitate the use of two-term recursions for the solution of the initial value problems. Therefore, the triangular second degree case with

$$\mathbf{F}_j = \mathbf{0} \quad j \in \{0, 1, \ldots\}; \quad j \neq 1, j \neq 2 \tag{11}$$

is taken as the main focus of this work. The resulting expansion is then $\mathbf{F}_1\mathbf{s}+\mathbf{F}_2\mathbf{s}^{\otimes 2}$, a multinomial shown by direct power. There is uniqueness for $\mathbf{F}_1\mathbf{s}$ unlike the arbitrariness in $\mathbf{F}_2\mathbf{s}^{\otimes 2}$. This is due to the nature of Kronecker power (which is specific form of the more abstract entity, direct power, over the ordinary linear algebraic entities).

The rest of the paper is organised as follows. The Sect. 2 covers certain details of the solution for the two block term recursion appearing in explicit autonomously conical ODEs. We revisit the separable kernel Kronecker power series and space extension to always get kernel separability (these are quite recently developed by the second author) in the Sects. 3 and 4. Section 5 involve the introduction of the telescopic matrices and the construction of the solution in terms of them, and, the characteristic initial vectors respectively; while the Sect. 6 completes the paper with concluding remarks.

2 Two block term recursions in the case of triangular conicality

The block ordinary differential equation involving the Kronecker powers of the basis set is as follows for the case where the evolution matrix has only two block diagonals, the main diagonal and its nearest upper neighbor

$$\dot{\mathbf{x}}_j(t) = \mathbf{E}_{j,j}\mathbf{x}_j + \mathbf{E}_{j,j+1}\mathbf{x}_{j+1}, \qquad \mathbf{x}_j(0) = \left(\mathbf{a}_{in} - \boldsymbol{\xi}^{(r)}\right)^{\otimes j} \qquad j = 0, 1, 2, \ldots \tag{12}$$

where the nonzero block elements of Evolution Matrix are explicitly given below

$$\begin{aligned} \mathbf{E}_{j,j} &\equiv \sum_{k=0}^{j-1} \mathbf{I}_n^{\otimes k} \otimes \mathbf{F}_1 \otimes \mathbf{I}_n^{\otimes j-k-1}, \\ \mathbf{E}_{j,j+1} &\equiv \sum_{k=0}^{j-1} \mathbf{I}_n^{\otimes k} \otimes \mathbf{F}_2 \otimes \mathbf{I}_n^{\otimes j-k-1}, \quad j = 0, 1, 2, \ldots \end{aligned} \tag{13}$$

each of which is apparently generated from a single matrix, the square matrix $\mathbf{F}_1$ and the rectangular matrix $\mathbf{F}_2$ respectively. Now the solution of (12) for $\mathbf{x}_j(t)$ can be written in the following new recursive form

$$\mathbf{x}_j(t) = \mathrm{e}^{t\mathbf{E}_{j,j}}\mathbf{x}_j(0) + \int_0^t d\tau \mathrm{e}^{(t-\tau)\mathbf{E}_{j,j}}\mathbf{E}_{j,j+1}\mathbf{x}_{j+1}(\tau), \quad j = 0, 1, 2, \ldots \tag{14}$$

Here, only the terms with subscripts j and $(j+1)$ appear as before. However, this is a two-term recursion involving integration hence it is free from the unbounded structure of the differentiation. This can be iterated once to get a relation between not $\mathbf{x}_j(t)$ and $\mathbf{x}_{j+1}(t)$ but $\mathbf{x}_j(t)$ and $\mathbf{x}_{j+2}(t)$. We obtain

$$
\begin{aligned}
\mathbf{x}_j(t) &= \mathrm{e}^{t\mathbf{E}_{j,j}}\mathbf{x}_j(0) \\
&\quad + \int_0^t d\tau \mathrm{e}^{(t-\tau)\mathbf{E}_{j,j}}\mathbf{E}_{j,j+1}\mathrm{e}^{\tau\mathbf{E}_{j+1,j+1}}\mathbf{x}_{j+1}(0) \\
&\quad + \int_0^t d\tau \mathrm{e}^{(t-\tau)\mathbf{E}_{j,j}}\mathbf{E}_{j,j+1}\int_0^\tau d\tau_1 \mathrm{e}^{(\tau-\tau_1)\mathbf{E}_{j+1,j+1}} \\
&\quad \times \mathbf{E}_{j+1,j+2}\mathbf{x}_{j+2}(\tau_1), \qquad j = 0, 1, 2, \ldots
\end{aligned} \tag{15}
$$

Since the summands of (13)'s right hand side are all commutative it is not hard to show that

$$
\mathrm{e}^{t\mathbf{E}_{j,j}} = \left\{ \mathrm{e}^{t\mathbf{F}_1} \right\}^{\otimes j}, \qquad j = 0, 1, 2, \ldots \tag{16}
$$

If we define

$$
\overline{\mathbf{E}}_{j,j+1}(\tau) \equiv \mathrm{e}^{-\tau\mathbf{F}_1}\mathbf{E}_{j,j+1}\left\{ \mathrm{e}^{\tau\mathbf{F}_1} \right\}^{\otimes 2}, \quad j = 0, 1, 2, \ldots \tag{17}
$$

then we can write

$$
\begin{aligned}
\mathbf{x}_j(t) &= \mathrm{e}^{t\mathbf{E}_{j,j}}\mathbf{x}_j(0) + \int_0^t d\tau \mathrm{e}^{t\mathbf{E}_{j,j}}\overline{\mathbf{E}}_{j,j+1}(\tau)\mathbf{x}_{j+1}(0) \\
&\quad + \int_0^t d\tau \mathrm{e}^{t\mathbf{E}_{j,j}}\overline{\mathbf{E}}_{j,j+1}(\tau)\int_0^\tau d\tau_1 \overline{\mathbf{E}}_{j+1,j+2}(\tau_1)\mathbf{x}_{j+2}(\tau_1), \quad j = 0, 1, 2, \ldots
\end{aligned} \tag{18}
$$

which can be simplified, by using the following integral operator definition over an arbitrary appropriate f

$$
\mathcal{I}_j\mathbf{f}(t) \equiv \int_0^t d\tau \overline{\mathbf{E}}_{j,j+1}(\tau)\mathbf{f}(\tau), \quad j = 0, 1, 2, \ldots, \tag{19}
$$

to the much more concise form

$$
\mathbf{x}_j(t) = \mathrm{e}^{t\mathbf{E}_{j,j}}\left[\mathbf{x}_j(0) + \mathcal{I}_j\mathbf{x}_{j+1}(0) + \mathcal{I}_j\mathcal{I}_{j+1}\mathbf{x}_{j+2}(t) \right], \quad j = 0, 1, 2, \ldots \tag{20}
$$

This is still recursion, not solution. To get solution we need to eliminate the unknowns from right hand side. One step iteration above created $\mathbf{x}_{j+2}(t)$ while removing $\mathbf{x}_{j+1}(t)$ at the right hand side. If we would realize two consecutive steps involving iteration then $\mathbf{x}_{j+3}(t)$ would be appearing while $\mathbf{x}_{j+1}(t)$ is removed at the right hand side. More generally, m number of consecutive steps including iteration creates $\mathbf{x}_{j+m+1}(t)$ while $\mathbf{x}_{j+1}(t)$ is removed at the right hand side. If m is taken to infinity and then $\mathbf{x}_{j+m+1}(t)$ is expected to be vanishing at that limit, the right hand side becomes unknown free. Even though the initial values appear at the right hand side they are known entities and therefore do not bring any unknown entity. All these allow us to write the following structure

$$\mathbf{x}_j(t) = e^{t\mathbf{E}_{j,j}} \left[\sum_{k=0}^{\infty} \left(\prod_{\ell=1}^{k} \mathcal{I}_{j+\ell-1} \right) \mathbf{x}_{j+k}(0) \right], \quad j = 0, 1, 2, \ldots \tag{21}$$

which is the formal solution for the jth Kronecker power of the system's state vector. Using the explicit structure of the initial vector $\mathbf{x}_j(0)$ we get

$$\mathbf{x}_j(t) = e^{t\mathbf{E}_{j,j}} \left[\sum_{k=0}^{\infty} \left(\prod_{\ell=1}^{k} \mathcal{I}_{j+\ell-1} \right) \mathbf{a}^{\otimes j+k} \right], \quad j = 0, 1, 2, \ldots \tag{22}$$

which becomes the solution of the original ODE set when j is taken as 1 and we can write

$$\boldsymbol{\xi}(t) \equiv \mathbf{x}_1(t) = e^{t\mathbf{F}_1} \left[\sum_{k=0}^{\infty} \left(\prod_{\ell=1}^{k} \mathcal{I}_{\ell} \right) \mathbf{a}^{\otimes k+1} \right]. \tag{23}$$

3 Rectangular commutativity and kernel separability in Kronecker power series

Although we have a Kronecker Power series in (23) its utilization has certain level complications which may increase the computational complexity. To reduce these negativities first we are going to seek a specific case where the temporal and rectangular matrix algebraic behavior of each summand in Kronecker power series can be factorized. The commutativity between square matrices presents many facilitations. Since $\mathbf{F}_2$ is rectangular while $\mathbf{F}_1$ is being square the multiplication between them is defined for their, only, one ordering if we use the ordinary matrix algebraic commutativity definition. However we can extend the definition (we call these equalities "Rectangular Commutativity) as follows

$$\mathbf{F}_2 \left(\mathbf{I}_n \otimes \mathbf{F}_1 \right) = \mathbf{F}_1 \mathbf{F}_2, \qquad \mathbf{F}_2 \left(\mathbf{F}_1 \otimes \mathbf{I}_n \right) = \mathbf{F}_1 \mathbf{F}_2 \tag{24}$$

which can be used to obtain

$$\begin{aligned}
\mathbf{F}_2 \left\{ \mathbf{I}_n \otimes e^{t\mathbf{F}_1} \right\} &= e^{t\mathbf{F}_1} \mathbf{F}_2, \\
\mathbf{F}_2 \left\{ e^{t\mathbf{F}_1} \otimes \mathbf{I}_n \right\} &= e^{t\mathbf{F}_1} \mathbf{F}_2, \\
\mathbf{F}_2 \left\{ e^{t_1\mathbf{F}_1} \otimes e^{t_2\mathbf{F}_1} \right\} &= e^{(t_1+t_2)\mathbf{F}_1} \mathbf{F}_2
\end{aligned} \tag{25}$$

where we have used the series representation of the exponential function together with certain properties of the Kronecker product.

All these permit us to obtain the following reductive equality

$$\overline{\mathbf{E}}_{1,2} = \overline{\mathbf{F}}_2(t) = \mathrm{e}^{t\mathbf{F_1}}\mathbf{F}_2, \tag{26}$$

and for more general purposes,

$$\overline{\mathbf{F}}_2\,(t_1)\,\mathrm{e}^{t_2(\mathbf{I}_n\otimes\mathbf{F}_1)+t_3(\mathbf{F}_1\otimes\mathbf{I}_n)} = \mathrm{e}^{(t_2+t_3-t_1)\mathbf{F}_1}\mathbf{F}_2. \tag{27}$$

The employment of these findings enables us to write the following explicit structures for the actions of the operators defined previously, by skipping the intermediate details

$$\mathcal{I}_1\mathbf{a}^{\otimes 2} = \left(\int_0^t d\tau \mathrm{e}^{\tau\mathbf{F}_1}\right)\mathbf{F}_2\mathbf{a}^{\otimes 2}, \tag{28}$$

$$\mathcal{I}_1\mathcal{I}_2\mathbf{a}^{\otimes 3} = \frac{1}{2!}\left(\int_0^t d\tau \mathrm{e}^{\tau\mathbf{F}_1}\right)^2 \mathbf{F}_2\left(\mathbf{I}_n\otimes\mathbf{F}_2+\mathbf{F}_2\otimes\mathbf{I}_n\right)\mathbf{a}^{\otimes 3}. \tag{29}$$

We report the following result for further generalization of these equalities by using the rectangular commutativity we have defined above.

$$\mathcal{I}_1\ldots\mathcal{I}_k\mathbf{a}^{\otimes k+1} = \frac{1}{k!}\left(\int_0^t d\tau \mathrm{e}^{\tau\mathbf{F}_1}\right)^k \mathbf{T}_k\mathbf{a}^{\otimes k+1} \tag{30}$$

where

$$\mathbf{T}_k \equiv \prod_{\ell=1}^{k}\mathbf{M}_\ell, \qquad k = 0, 1, 2, \ldots \tag{31}$$

and

$$\mathbf{M}_j \equiv \sum_{k=0}^{j-1}\mathbf{I}_n^{\otimes k}\otimes\mathbf{F}_2\otimes\mathbf{I}_n^{\otimes j-1-k}. \tag{32}$$

Everywhere in this formulation we have followed the universal convention which dictates us that a sum and product is taken equal to 0 and 1 respectively without regarding to their summands and factors when their upper limits are less than the lower one.

Now we can rewrite (23) as follows (where $\mathbf{T}_0$ is an appropriate type identity matrix)

$$\boldsymbol{\xi}(t) = \mathrm{e}^{t\mathbf{F}_1}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\int_0^t d\tau \mathrm{e}^{\tau\mathbf{F}_1}\right)^j \mathbf{T}_j\mathbf{a}^{\otimes j+1} \tag{33}$$

which is an infinite sum over the temporal entities, jth of which is defined as the jth power of the integral of the exponential matrix $e^{t\mathbf{F}_1}$ between 0 and t, divided by $j!$. This term contains not only temporal change but also square matrix character as algebraic behavior. The jth term of this sum also contains $\mathbf{T}_j$ which is completely a matrix algebraic entity. As we know from above, $\mathbf{T}_j$ is the product of horizontally rectangular matrices $\mathbf{M}$s. By definition $\mathbf{M}_j$ is of $n^j \times n^{j+1}$ type. It maps from the n^{j+1} dimensional Cartesian space to n^j dimensional Cartesian space. In $\mathbf{T}_j\mathbf{a}^{\otimes j+1}$ term the image of $\mathbf{a}^{\otimes j+1}$ under $\mathbf{T}_j$ is considered. However this image is created through a consecutive imaging process. First, the image of $\mathbf{a}^{\otimes j+1}$ under $\mathbf{M}_j$ is created. This image lies in the n^j dimensional Cartesian space and its image under $\mathbf{M}_{j-1}$ is created in the n^{j-1} dimensional Cartesian space and so on. If we consider that the n^j dimensional Cartesian spaces are ordered as if from close to far distances in ascending j values then $\mathbf{M}$ matrices can be considered scoping from one Cartesian space to its first lower dimensional one (from n^j dimension to n^{j-1} dimension), that is, create the images of the images, as if from far distance to close distance. Thus the entire effect of the matrix $\mathbf{T}_j$ on $\mathbf{a}^{\otimes j+1}$ can be considered as scoping from n^{j+1} dimensional Cartesian space to the nearest space, n dimensional Cartesian space. For this reason we call these matrices “Telescope (or Telescopic) Matrices”. All these mean that the rectangular matrix algebraic factor of the infinite sum’s kernel in (33) carries in fact a telescoping nature mapping from higher dimensions to n dimension.

4 Most general and efficient kernel separability

Thus we have shown that the kernel (summand) of the Kronecker power series solution can be factorized to a temporally varying square matrix and a time invariant rectangular matrix structure if the matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ are rectangularly commutative. This however, truly speaking, condenses the temporal change into a single factor. The matrix algebraic nature still exists in both factors of the binary product type kernel, square matrix type for temporally varying factor and rectangular matrix for temporally invariant factor. The rectangularity in matrix natures is separated out from the squareness.

Let us now consider the following most general form of the conical explicit ODE sets in Kronecker power series representation

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{F}_0 + \mathbf{F}_1\boldsymbol{\xi}(t) + \mathbf{F}_2\boldsymbol{\xi}(t)^{\otimes 2}, \qquad \boldsymbol{\xi}(0) = \mathbf{a}. \tag{34}$$

We can extend the space spanned by the vector $\boldsymbol{\xi}(t)$’s by using the following augmented unknown vector

$$\mathbf{x}_{aug}(t) \equiv \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \\ \xi_{n+1}(t) \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\xi}(t) \\ \xi_{n+1}(t) \end{bmatrix} \tag{35}$$

where $\xi_{n+1}(t)$ can be anything in principle. However, we specifically choose it a temporally invariant entity, a constant in time. The constancy means that its initial value will remain as its values for all time instances. To get a typographical harmony in the formulation we will denote its initial value by $a_{n+1}^{(in)}$.

In order to facilitate the analysis, permutation matrix

$$\mathbf{P}_2 = \begin{bmatrix} \boldsymbol{\pi}_1^T \\ \boldsymbol{\pi}_2^T \\ \vdots \\ \boldsymbol{\pi}_{(n+1)^2}^T \end{bmatrix}, \tag{36}$$

$$\boldsymbol{\pi}_i = \begin{cases} \mathbf{e}_{n^2+i/(n+1)} & \text{if } i = kn+k, k = 1, \ldots, n \\ \mathbf{e}_{i-\lfloor i/(n+1)\rfloor} & \text{if } i \neq kn+k, k = 1, \ldots, n \text{ and } i < n^2+n \\ \mathbf{e}_i & \text{if } i \geq n^2+n \end{cases} \tag{37}$$

is utilized. $\mathbf{e}_i$ stands for the ith Cartesian unit vector of $(n+1)^2$ dimensional space and has 1 as the only nonzero element located at the ith position. $\mathbf{P}_2$ is an $(n+1)^2 \times (n+1)^2$ matrix. It facilitates the use of the vector with blocks $\boldsymbol{\xi}(t)^{\otimes 2}$, $\boldsymbol{\xi}(t)\xi_{n+1}(t)$, $\xi_{n+1}(t)\boldsymbol{\xi}(t)$ and $\xi_{n+1}(t)^2$ respectively instead of $\mathbf{x}_{aug}(t)^{\otimes 2}$ so that the multiplied blocks may be seen. Consequently,

$$\begin{bmatrix} \boldsymbol{\xi}(t)^{\otimes 2} \\ \boldsymbol{\xi}(t)\xi_{n+1}(t) \\ \xi_{n+1}(t)\boldsymbol{\xi}(t) \\ \xi_{n+1}(t)^2 \end{bmatrix} = \mathbf{P}_2^{-1} \begin{bmatrix} \boldsymbol{\xi}(t) \\ \xi_{n+1}(t) \end{bmatrix}^{\otimes 2} \tag{38}$$

holds. By differentiating both sides of (35) with respect to time and then using (34) we can write

$$\dot{\mathbf{x}}_{aug}(t) = \mathbf{F}_1^{(aug)}\mathbf{x}_{aug}(t) + \mathbf{F}_2^{(aug)}\mathbf{x}_{aug}(t)^{\otimes 2}, \qquad \mathbf{x}_{aug}(0) = \mathbf{a}_{aug} \equiv \begin{bmatrix} \mathbf{a}_{in} \\ a_{n+1}^{(in)} \end{bmatrix} \tag{39}$$

where

$$\mathbf{F}_1^{(aug)} \equiv \begin{bmatrix} \mathbf{F}_1 & \frac{1}{a_{n+1}^{(in)}}\mathbf{F}_0 \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \qquad \mathbf{F}_2^{(aug)} \equiv \begin{bmatrix} \mathbf{F}_2 & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n^2} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 0 \end{bmatrix} \mathbf{P}_2^{-1} \tag{40}$$

In these formulae $\mathbf{0}$s identify the zero matrices whose types are shown in their subindices. These equations do not give the final forms of the augmented $\mathbf{F}_1$ and $\mathbf{F}_2$ matrices. To explain this we can define

$$\mathbf{u}_j \equiv \begin{bmatrix} \mathbf{0}_{n^2\times 1} \\ \mathbf{e}_j \\ \mathbf{0}_{n\times 1} \\ 0 \end{bmatrix}, \quad \mathbf{u}_{n+j} \equiv \begin{bmatrix} \mathbf{0}_{n^2\times 1} \\ \mathbf{0}_{n\times 1} \\ \mathbf{e}_j \\ 0 \end{bmatrix}, \mathbf{u}_{2n+1} \equiv \begin{bmatrix} \mathbf{0}_{n^2\times 1} \\ \mathbf{0}_{n\times 1} \\ \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix}; \quad j = 1, 2, \ldots, n; \tag{41}$$

$$\mathbf{v}_j \equiv \begin{bmatrix} \mathbf{e}_j \\ 0 \end{bmatrix}, \quad j = 1, 2, \ldots, n, \quad \mathbf{v}_{n+1} \equiv \begin{bmatrix} \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix} \tag{42}$$

where $\mathbf{e}_j$ stands for the jth Cartesian unit vector. These permit us to write the following $(2n + 1)$ number of scalar identities over the augmented unknown vector and its Kronecker square

$$\begin{aligned}
&\mathbf{u}_{2n+1}^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_{n+1}^T \mathbf{x}_{aug}(t) = 0, \\
&\mathbf{u}_j^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_j^T \mathbf{x}_{aug}(t) = 0, \\
&\mathbf{u}_{n+j}^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_j^T \mathbf{x}_{aug}(t) = 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{43}$$

These urge us to produce the following $(n + 1)(2n + 1)$ number of vector identities

$$\begin{aligned}
&\mathbf{v}_i \mathbf{u}_{2n+1}^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_i \mathbf{v}_{n+1}^T \mathbf{x}_{aug}(t) = \mathbf{0}_{(n+1)\times 1}, \\
&\mathbf{v}_i \mathbf{u}_j^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_i \mathbf{v}_j^T \mathbf{x}_{aug}(t) = \mathbf{0}_{(n+1)\times 1}, \\
&\mathbf{v}_i \mathbf{u}_{n+j}^T \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} - \xi_{n+1} \mathbf{v}_i \mathbf{v}_j^T \mathbf{x}_{aug}(t) = \mathbf{0}_{(n+1)\times 1}, \\
&i = 1, \ldots, (n + 1); \;\; j = 1, 2, \ldots, n
\end{aligned} \tag{44}$$

whose left hand side expressions can be added to the right hand side of (39) without causing any alteration. Thus we can write

$$\begin{aligned}
&\dot{\mathbf{x}}_{aug}(t) = \mathbf{F}_1^{(faug)} \mathbf{x}_{aug}(t) + \mathbf{F}_2^{(faug)} \mathbf{x}_{aug}(t)^{\otimes 2}, \\
&\mathbf{x}_{aug}(0) = \mathbf{a}_{aug} \equiv \begin{bmatrix} \mathbf{a}_{in} \\ a_{n+1}^{(in)} \end{bmatrix}
\end{aligned} \tag{45}$$

where the superscript $(faug)$ stands as an abbreviation for the statement "flexible augmented" and

$$\mathbf{F}_1^{(faug)} \equiv \mathbf{F}_1^{(aug)} + a_{n+1}^{(in)} \sum_{i=1}^{n+1} \left(b_{i,2n+1} \mathbf{v}_i \mathbf{v}_{n+1}^T + \sum_{j=1}^{n} \left(b_{i,j} + b_{i,n+j} \right) \mathbf{v}_i \mathbf{v}_j^T \right),$$

$$\mathbf{F}_2^{(faug)} \equiv \mathbf{F}_2^{(aug)} - \left(\sum_{i=1}^{n+1} b_{i,2n+1} \mathbf{v}_i \mathbf{u}_{2n+1}^T + \sum_{j=1}^{n} \left(b_{i,j} \mathbf{v}_i \mathbf{u}_j^T + b_{i,n+j} \mathbf{v}_i \mathbf{u}_{n+j}^T \right) \right) \mathbf{P}_2^{-1}. \tag{46}$$

We have used the fact that $\xi_{n+1} \equiv a_{n+1}^{(in)}$ in the formulation of these equalities. bs in last two equalities stand for arbitrary parameters at this moment. We can use the following impositions for arbitrary β parameter values to determine bs

$$\mathbf{F}_1^{(faug)} \equiv -\beta \mathbf{I}_{n+1} \tag{47}$$

where the minus sign at the right hand side is used to get some convenience for our future needs. This means

$$b_{n+1,2n+1} = -\frac{\beta}{a_{n+1}^{(in)}}, \quad b_{i,2n+1} = -\frac{[\,\mathbf{F}_0\,]_i}{a_{n+1}^{(in)}}, \quad b_{n+1,n+i} = -b_{n+1,i}, \quad i = 1, 2, \ldots, n,$$

$$b_{i,n+j} = -\frac{\beta}{a_{n+1}^{(in)}} \delta_{i,j} - \frac{[\,\mathbf{F}_1\,]_{i,j}}{a_{n+1}^{(in)}} - b_{i,j}, \qquad i, j = 1, 2, \ldots, n \tag{48}$$

where subscripted square brackets notation is used for obtaining the corresponding element from the vector or matrix under consideration. Therefore, flexible augmented matrix coefficient is

$$\begin{aligned}
\mathbf{F}_2^{(faug)} = \mathbf{F}_2^{(aug)} + \Bigg(& \frac{\beta}{a_{n+1}^{(in)}} \mathbf{v}_{n+1} \mathbf{u}_{2n+1}^T + \sum_{i=1}^{n} \frac{[\,\mathbf{F}_0\,]_i}{a_{n+1}^{(in)}} \mathbf{v}_i \mathbf{u}_{2n+1}^T \\
& - \sum_{j=1}^{n} b_{n+1,j} \mathbf{v}_{n+1} \left(\mathbf{u}_j^T - \mathbf{u}_{n+j}^T \right) - \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} \mathbf{v}_i \left(\mathbf{u}_j^T - \mathbf{u}_{n+j}^T \right) \\
& + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{[\,\mathbf{F}_1\,]_{i,j}}{a_{n+1}^{(in)}} \mathbf{v}_i \mathbf{u}_{n+j}^T + \frac{\beta}{a_{n+1}^{(in)}} \sum_{i=1}^{n} \mathbf{v}_i \mathbf{u}_{n+i}^T \Bigg) \mathbf{P}_2^{-1}.
\end{aligned} \tag{49}$$

A careful investigation reveals the validities of the following equalities

$$\left(\mathbf{u}_j^T - \mathbf{u}_{n+j}^T \right) \mathbf{P}_2^{-1} \mathbf{x}_{aug}(t)^{\otimes 2} = 0, \qquad j = 1, 2, \ldots, n \tag{50}$$

which enable us to exclude all terms including left hand side expressions from the formulation and then (49) becomes

$$\mathbf{F}_2^{(faug)} = \mathbf{F}_2^{(aug)} + \left(\frac{\beta}{a_{n+1}^{(in)}} \mathbf{v}_{n+1} \mathbf{u}_{2n+1}^T + \sum_{i=1}^{n} \frac{[\,\mathbf{F}_0\,]_i}{a_{n+1}^{(in)}} \mathbf{v}_i \mathbf{u}_{2n+1}^T \right.$$
$$\left. + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{[\,\mathbf{F}_1\,]_{i,j}}{a_{n+1}^{(in)}} \mathbf{v}_i \mathbf{u}_{n+j}^T + \frac{\beta}{a_{n+1}^{(in)}} \sum_{i=1}^{n} \mathbf{v}_i \mathbf{u}_{n+i}^T \right) \mathbf{P}_2^{-1}. \tag{51}$$

Thus the matrix $\mathbf{F}_2^{(faug)}$ now contains just a single arbitrary parameter, β, only. That parameter can be used to provide us with certain properties in the formulation we want. (47) and (51) are now the new matrix coefficients of the extended space representation of the original ODE set.

The use of (47) and (51) urges us to write

$$\mathbf{x}(t) = \mathrm{e}^{-\beta t} \sum_{j=0}^{\infty} \left[\prod_{k=1}^{j} \widehat{\mathbf{J}}_k \right] \mathbf{a}_{in}^{\otimes j+1}, \tag{52}$$

$$\widehat{\mathbf{J}}_j \mathbf{g}(t) \equiv \int_0^t d\tau \mathrm{e}^{-\beta\tau} \left(\sum_{k=0}^{j-1} \mathbf{I}_n^{\otimes k} \otimes \mathbf{F}_2^{(faug)} \otimes \mathbf{I}_n^{\otimes j-1-k} \right) \mathbf{g}(\tau) \tag{53}$$

where $\mathbf{g}(t)$ is any vector valued dimensionally compatible temporal function. This implies

$$\widehat{\mathbf{J}}_1 \ldots \widehat{\mathbf{J}}_k \mathbf{a}^{\otimes k+1} = \frac{1}{k!} \left(\frac{1-\mathrm{e}^{-\beta t}}{\beta} \right)^k \mathbf{T}_k \mathbf{a}^{\otimes k+1} \tag{54}$$

where

$$\mathbf{T}_k \equiv \prod_{\ell=1}^{k} \mathbf{M}_\ell, \qquad k = 0, 1, 2, \ldots \tag{55}$$

and

$$\mathbf{M}_j \equiv \sum_{k=0}^{j-1} \mathbf{I}_n^{\otimes k} \otimes \mathbf{F}_2^{(faug)} \otimes \mathbf{I}_n^{\otimes j-1-k} \tag{56}$$

and therefore

$$\mathbf{x}(t) = \mathrm{e}^{-\beta t} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1-\mathrm{e}^{-\beta t}}{\beta} \right)^j \mathbf{T}_j \mathbf{a}^{\otimes j+1}. \tag{57}$$

5 Rectangular eigenvalue problem

(57) is a Kronecker power series whose kernel (or summand) has a scalar prefactor which can temporally vary through an exponential function including function structure. This prefactor multiplies an n element vector which is the image of a Kronecker power of the initial vector under the corresponding rectangular $\mathbf{T}$ matrix. Thus, the temporal and matrix algebraic structures become now truly separated. However, this separation is not the ultimate level of simplification. To get more simplicity we may focus on the following equation.

$$\mathbf{F}_2^{(faug)}\boldsymbol{\phi}^{\otimes 2} = \varphi\boldsymbol{\phi}. \tag{58}$$

If this equation has at least one solution, that is, one couple of φ scalar and $\boldsymbol{\phi}$ vector values then we can write

$$\mathbf{M}_k\boldsymbol{\phi}^{\otimes k+1} = k\varphi\boldsymbol{\phi}^{\otimes k}, \qquad \mathbf{T}_k\boldsymbol{\phi}^{\otimes k+1} = k!\varphi^k\boldsymbol{\phi}, \tag{59}$$

$$\widehat{\mathbf{J}}_1\ldots\widehat{\mathbf{J}}_k\boldsymbol{\phi}^{\otimes k+1} = \left(\frac{1-\mathrm{e}^{-\beta t}}{\beta}\right)^k\varphi^k\boldsymbol{\phi}, \tag{60}$$

$$\mathbf{x}(t) = \frac{\mathrm{e}^{-\beta t}}{1-\left(\frac{1-\mathrm{e}^{-\beta t}}{\beta}\right)\varphi}\boldsymbol{\phi}. \tag{61}$$

Thus the solution of the original ODEs can be given through a concise formula as long as the initial vector is $\boldsymbol{\phi}$.

Now we can investigate the existence of φ and $\boldsymbol{\phi}$. To this end we can start with the following equality

$$\mathbf{F}_2^{(faug)} \equiv \begin{bmatrix} \mathbf{F}_2 & \mathbf{0}_{n\times n} & \frac{1}{a_{n+1}^{(in)}}\left(\mathbf{F}_1+\beta\mathbf{I}_n\right) & \frac{1}{a_{n+1}^{(in)}}\mathbf{F}_0 \\ \mathbf{0}_{1\times n^2} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \frac{\beta}{a_{n+1}^{(in)}} \end{bmatrix}\mathbf{P}_2^{-1} \tag{62}$$

whose utilization in (58) allows us to get the following equations

$$\mathbf{F}_2\overline{\boldsymbol{\phi}}^{\otimes 2} + \frac{\overline{\phi}_{n+1}}{a_{n+1}^{(in)}}\left(\mathbf{F}_1+\beta\mathbf{I}_n\right)\overline{\boldsymbol{\phi}} + \frac{\overline{\phi}_{n+1}^2}{a_{n+1}^{(in)}}\mathbf{F}_0 = \varphi\overline{\boldsymbol{\phi}}\,\frac{\overline{\phi}_{n+1}^2}{a_{n+1}^{(in)}}\beta = \varphi\overline{\phi}_{n+1} \tag{63}$$

where we have used the following partitioning

$$\boldsymbol{\phi} \equiv \begin{bmatrix} \overline{\boldsymbol{\phi}} \\ \overline{\phi}_{n+1} \end{bmatrix}. \tag{64}$$

The $(n+1)$th element of the reigenvector should be same as the value of the constancy used in CASE. Hence we need to take

$$\overline{\phi}_{n+1} = a_{n+1}^{(in)} \tag{65}$$

which implies

$$\varphi = \beta \tag{66}$$

and therefore

$$\mathbf{F}_2 \overline{\boldsymbol{\phi}}^{\otimes 2} + \mathbf{F}_1 \overline{\boldsymbol{\phi}} + a_{n+1}^{(in)} \mathbf{F}_0 = \mathbf{0}_{n\times 1}. \tag{67}$$

The last formula defines an algebraic root finding problem of a set of second degree multinomials (n number of equations for n unknowns). The existence of the roots, the number of roots if exist, and, the methods for how to determine them are all related to the set theoretical and functional analysis based items "ideals". To this end, well known and widely used theory is based on Gröbner basis set and the Buchberger algorithm for practical applications. The roots, one or more than one set of element values for $\overline{\phi}_j$s $(j = 1, 2, \ldots, n)$ may or may not have real values. However the solution family is not empty if the complex number values are considered. Hence, the reigenvalue problem (rectangular eigenvalue problem) above has at least one solution which describes a position in the space spanned by $\overline{\phi}_j$s $(j = 1, 2, \ldots, n)$. This position depends on $a_{n+1}^{(in)}$ or its equivalent $\overline{\phi}_{n+1}$ as long as the vector $\mathbf{F}_0$ remains nonvanishing. Otherwise the point described by the reigenvector remains standing at the same position of $\overline{\phi}_j$s $(j = 1, 2, \ldots, n)$ for all these parameters' values. All these mean that reigenvector determined as the root above equations remains on a straight line where $\overline{\phi}_{n+1}$ does not affect the position in the space spanned by $\overline{\phi}_j$s $(j = 1, 2, \ldots, n+2)$, or otherwise, a curve whose shape is completely determined by $\overline{\phi}_{n+1}$ in the same space spanned by $\overline{\phi}_j$s $(j = 1, 2, \ldots, n+2)$. We call these lines or curves "Reigen Curves". Thus, the facilitations coming from the rectangular eigenvalue problem solutions will be in action if and only if the initial vector of the original ODEs is positioned at somewhere on the reigencurve.

Now the utilization of (66) in (61) takes us to the following very simple result

$$\mathbf{x}(t) = \boldsymbol{\phi} \tag{68}$$

which dictates us that the original ODE set has a constant solution if the initial value vector in its accompanying imposition is positioned at somewhere on one of reigencurves. This result could be obtained directly from the original ODE set without proceeding in the jungle of so many details of the analysis. However this analysis indicated that kernel separation is an important issue which may bring unexpectedly good facilitations.

6 Conclusion

This has been the second part of two companion papers. In addition to the new findings and ideas presented in the first part we have arrived at certain important points. However, despite what we have developed here reveals many important features existing in the application of Kronecker power series to explicit conical ODEs, they are not at the end point of the probabilistic evolution theory and the related issues. On the contrary, they may be considered as the beginnings of new revelations or theories about the ODEs. We enumerate the important points and remarks below:

1. We have obtained the analytic form of the solution for the initial value problem of a finite number of explicit conical ODEs. We now know that the solution can be written in a Kronecker power series form where the summand has analytical expression.
2. The summand or (additive) kernel of the solution series can be simplified if the coefficient matrices of the originally given ODE set satisfy certain commutativity (rectangular commutativity) relations. If this happens then the temporal behavior of the kernel is condensed in the first factor of a binary product which has also square matrix structure. The second factor does not contain the time and shows rectangular matrix structure whose number of columns increases as we proceed amongst the summands of the relevant Kronecker power series in ascending power direction.
3. We know that the addition of a constant temporal function to the unknowns as a new member and therefore increase in the dimension of the unknowns' space by one, enable us to get rid of the constant matrix component ($\mathbf{F}_0$). Beyond that, by using the flexibilities appearing in the resulting matrix structures, we can change the first degree terms coefficient matrix to an identity matrix premultiplied by an arbitrary constant (β). This structuring facilitates our analysis pretty much and separates the matrix algebraic nature from the temporal behavior in the kernel, by transferring the squareness to the rectangularity. This is the full separation of matrix algebra and temporal change.
4. We now know that the full kernel separability takes us to the constant solution of the original ODEs if we use the so-called "Rectangular Eigenvalue Problem". This also urges us to define rectangular eigencurves or shortly reigencurves.
5. Even though we have not intended to go beyond the constant solution, what we have obtained here implies that we can possibly obtain different specific structure solutions by tracing the route we followed when we get the constant solution within certain level of deviations in methodology. We started to work on these issues, to get new horizons in the ODE theory.
6. We do not need to get constant solution and we have the expressions to get the solutions without assuming any rectangular eigen structure. The algorithm seems to be simple conceptually while certain precautions should be taken for the practical evaluations in the sense of computation time and the memory utilization in computers.

Before finalizing these two companion works presentation, what we need to emphasize on is that this second paper content includes the techniques to extend the capabilities of the probabilistic evolution approach and to remove possible pitfalls which may negatively affect the numerical qualities of the truncation approximants.

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